Constructing Invariant Tori for Two Weakly Coupled van der Pol Oscillators *

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Abstract. An algorithm is developed for the construction of an invariant torus of a weakly coupled autonomous oscillator. The system is put into angular standard form. The determining equations are found by averaging and are solved for the approximate amplitudes of the torus. A perturbation series is then constructed about the approximate amplitudes with unknown coefficients as periodic functions of the angular variables. A sequence of solvable partial differential equations is developed for determining the coefficients. The algorithm is applied to a system of nonlinearly coupled van der Pol equations and the first order coefficients are generated in a straightforward manner. The approximation shows both good numerical accuracy and reproducibility of the periodicities of the van der Pol system. A comparitive analysis of integrating the van der Pol system with integrating the phase equations from the angular standard form on the approximate torus shows numerical errors of the order of the perturbation parameter $\epsilon = 0.05$ for integrations of up to 10,000 steps. Applying FFT to the numerical periodicities generated by integrating the van der Pol system near the torus reveals the same predominant frequencies found in the perturbation coefficients. Finally an expected rotation number is found by integrating the phase equations on the approximate torus.

Key words: Averaging, invariant torus, nonlinear oscillator, van der Pol oscillator.

1. Introduction

In this work an algorithm for the direct construction of the invariant tori for a pair of weakly nonlinearly coupled van der Pol oscillators is developed. The system under study is given by

$$\begin{aligned} \ddot{z}_1 + \mu_1^2 z_1 &= \epsilon(1 - z_1^2 - a z_2^2)\dot{z}_1, \\ \ddot{z}_2 + \mu_2^2 z_2 &= \epsilon(1 - \alpha z_1^2 - z_2^2)\dot{z}_2, \end{aligned} \tag{1}$$

where $a > 0$, $\alpha > 0$ and $\mu_1 > 0$, and $\mu_2 > 0$ are linearly independent over the integers. Equations (1) have been previously studied in connection with generalized averaging methods and integral manifolds by Hale [1] and Gilsinn [2,3]. The algorithm developed in this work involves a perturbation technique that yields a sequence of solvable first-order partial differential equations for computing the coefficients. The technique is motivated by a perturbation procedure used by Minorsky [4] to develop an integral curve representation for the periodic solution of a single van der Pol equation.

The dynamics arising from the interaction of coupled van der Pol oscillators have been studied by many authors in the areas of engineering, electronics, high energy physics and biology. For example, they have been studied by Minorsky [4] in coupled oscillating circuits, by Hall and Iwan [5] in vortex-induced oscillations, Hofmann and Jowett [6] and Jowett [7] in synchroton and horizontal betatron oscillations, Rand and Holmes [8] in a study of phase-locked periodic motions, Storti and Rand [9] in a study of steady state behavior with strong linear diffusive coupling [9] also provides a bibliography with further references to studies of coupled van der Pol oscillations.

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Several authors have studied methods of approximating tori. Diliberto *et al.* [10] used a method based on the existence of first integrals. Mitropolskii [11] developed a procedure related to the generalized method of averaging to approximate some classes of multi-parameter families of integral curves. Ioos and Joseph [12] developed approximations for the cross-section of the torus. Gilsinn [13] extended an approximation result given in Carr [14] for center manifolds to integral manifolds which become tori in the autonomous case. Dieci *et al.* [15] have recently developed a numerical algorithm to compute invariant tori by solving the partial differential equation satisfied by the tori, but does not yield a representation of the tori. The method developed here differs from the previous methods in that it is a direct perturbation technique for approximating a torus. It develops a representation formula that reveals the character of the various periodicities of the torus. The method is similar to Whittaker's [16] adelphic integral method for computing integrals of motion for conservative systems. Both approaches develop sequences of solvable partial differential equations to determine the perturbation coefficients for the expansion. The perturbation technique developed in this paper, however, is applicable to nonconservative systems.

2. Perturbation Algorithm

If we set

$$\Omega = \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{bmatrix}, \tag{2}$$

then equation (1) can be written in the form

$$\ddot{\boldsymbol{z}} + \Omega^2 \boldsymbol{z} = \epsilon \boldsymbol{Z}(\boldsymbol{z}, \dot{\boldsymbol{z}}) \tag{3}$$

where $\boldsymbol{z} = (z_1, z_2)^T$, $\dot{\boldsymbol{z}} = (\dot{z}_1, \dot{z}_2)^T$, $\ddot{\boldsymbol{z}} = (\ddot{z}_1, \ddot{z}_2)^T$ and $\boldsymbol{Z} = (Z_1, Z_2)^T$. The parameters a and α have been eliminated for simplicity. Bold characters are used to represent vectors.

Introduce coordinates $\boldsymbol{r} = (r_1, r_2)^T$ and $\boldsymbol{\phi} = (\phi_1, \phi_2)^T$ into equation (3) by the transformation

$$\begin{aligned} z_j &= r_j^\gamma \sin \mu_j \phi_j, \\ \dot{z}_j &= \mu_j r_j^\gamma \cos \mu_j \phi_j, \end{aligned} \tag{4}$$

where $r_j \geq 0$, $j = 1, 2$. The parameter $\gamma > 0$ is introduced in order to simplify expressions if necessary. The new system becomes

$$\begin{aligned} \dot{\boldsymbol{\phi}} &= \boldsymbol{d} + \epsilon \boldsymbol{\Phi}(\boldsymbol{\phi}, \boldsymbol{r}), \\ \dot{\boldsymbol{r}} &= \epsilon \boldsymbol{R}(\boldsymbol{\phi}, \boldsymbol{r}), \end{aligned} \tag{5}$$

where $\boldsymbol{d} = (1, 1)^T$, $\boldsymbol{\Phi} = (\Phi_1, \Phi_2)^T$, $\boldsymbol{R} = (R_1, R_2)^T$ and

$$\begin{aligned} \Phi_j(\boldsymbol{\phi}, \boldsymbol{r}) &= -(1/r_j^\gamma \mu_j^2) Z_j(\boldsymbol{z}, \dot{\boldsymbol{z}}) \sin \mu_j \phi_j \\ R_j(\boldsymbol{\phi}, \boldsymbol{r}) &= (1/r_j^{\gamma-1} \mu_j^2) Z_j(\boldsymbol{z}, \dot{\boldsymbol{z}}) \cos \mu_j \phi_j, \end{aligned} \tag{6}$$

for $j = 1, 2$.

Define the average of $\boldsymbol{R}(\boldsymbol{\phi}, \boldsymbol{r})$ from equation (5) by

$$\boldsymbol{R}_0(\boldsymbol{r}) = (\mu_1 \mu_2 / 4\pi^2) \int_0^{2\pi/\mu_1} \int_0^{2\pi/\mu_2} \boldsymbol{R}(\boldsymbol{\phi}, \boldsymbol{r}) \, \mathrm{d}\phi_1 \, \mathrm{d}\phi_2, \tag{7}$$

where $\boldsymbol{R}_0(R_{01}, R_{02})^T$. $\boldsymbol{R}_0$ is independent of $\boldsymbol{\phi}$ since μ_1, μ_2 are linearly independent over the integers.

In order to determine the existence of tori a result of Hale [1] on the existence of integral manifolds can be applied to equation (5). Let

$$\frac{\partial \boldsymbol{R}_0}{\partial \boldsymbol{r}} = \begin{bmatrix} \dfrac{\partial R_{01}}{\partial r_1} & \dfrac{\partial R_{01}}{\partial r_2} \\ \dfrac{\partial R_{02}}{\partial r_1} & \dfrac{\partial R_{02}}{\partial r_2} \end{bmatrix}$$

$\boldsymbol{\Phi}$ and $\boldsymbol{R}$ in equation (5) are periodic in $\boldsymbol{\phi}$ with vector period $(2\pi/\mu_1, 2\pi/\mu_2)^T$ in $(\phi_1, \phi_2)^T$. Let $\boldsymbol{r}^{(0)}$ be a vector such that $\boldsymbol{R}_0(\boldsymbol{r}^{(0)}) = \mathbf{0}$ and the real parts of the eigenvectors of $\partial \boldsymbol{R}_0/\partial \boldsymbol{r}$ have nonzero real parts. Then there is a vector function $\boldsymbol{r}(\boldsymbol{\phi}, \epsilon)$, periodic in $\boldsymbol{\phi}$ with vector period $(2\pi/\mu_1, 2\pi/\mu_2)^T$, $\boldsymbol{r}(\boldsymbol{\phi}, 0) = \boldsymbol{r}^{(0)}$ and $\boldsymbol{r}(\boldsymbol{\phi}, \epsilon)$ is an integral manifold for equation (5) or

$$\begin{aligned} z_j &= r_j(\boldsymbol{\phi}, \epsilon)^\gamma \sin \mu_j \phi_j, \\ \dot{z}_j &= \mu_j r_j(\boldsymbol{\phi}, \epsilon)^\gamma \cos \mu_j \phi_j, \end{aligned} \tag{8}$$

$j = 1, 2$, is an integral manifold for equation (3). Since $\boldsymbol{r}(\boldsymbol{\phi}, \epsilon)$ is independent of t it has been called a periodic surface by Diliberto and his colleagues [10]. By imbedding $\boldsymbol{r}(\boldsymbol{\phi}, \epsilon)$ in R^3 by the relations

$$\begin{aligned} x &= (r_1 + r_2 \cos \mu_2 \phi_2) \cos \mu_1 \phi_1, \\ y &= (r_1 + r_2 \cos \mu_2 \phi_2) \sin \mu_1 \phi_1, \\ z &= r_2 \sin \mu_2 \phi_2, \end{aligned} \tag{9}$$

for $0 \leq \phi_1 < 2\pi/\mu_1$, $0 \leq \phi_2 < 2\pi/\mu_2$, $0 < r_1 < r_2$, the surface $\boldsymbol{r}(\boldsymbol{\phi}, \epsilon)$ becomes a torus. The solutions on the torus are determined by solving

$$\dot{\boldsymbol{\phi}} = \boldsymbol{d} + \epsilon \boldsymbol{\Phi}(\boldsymbol{\phi}, \boldsymbol{r}(\boldsymbol{\phi}, \epsilon)) \tag{10}$$

for $\boldsymbol{\phi}$ then substituting it into $\boldsymbol{r}(\boldsymbol{\phi}, \epsilon)$. The stability of the periodic surface is locally the same as the stability properties of the $\boldsymbol{w} = \mathbf{0}$ solution of the system

$$\dot{\boldsymbol{w}} = \frac{\partial \boldsymbol{R}_0}{\partial \boldsymbol{r}}(\boldsymbol{r}^{(0)})\boldsymbol{w}.$$

For a further discussion see Hale [1]. The principal objective in this work is not stability questions but to develop a constructive procedure to approximate $\boldsymbol{r}(\boldsymbol{\phi}, \epsilon)$.

To begin with the construction assume an expansion for $\boldsymbol{r} = \boldsymbol{r}(\boldsymbol{\phi}, \epsilon)$ in the form

$$\boldsymbol{r} = \sum_{n=0}^{\infty} \epsilon^n \boldsymbol{r}^{(n)}(\boldsymbol{\phi}), \tag{11}$$

where we seek $\boldsymbol{r}^{(n)}(\boldsymbol{\phi})$, $n = 0, 1, \cdots$, each with vector period $(2\pi/\mu_1, 2\pi/\mu_2)$. In terms of coordinates $\boldsymbol{r}^{(n)}(\boldsymbol{\phi}) = (r_1^{(n)}(\boldsymbol{\phi}), r_2^{(n)}(\boldsymbol{\phi}))^T$ for each n.

Since we assume that equation (11) is a periodic surface, it must satisfy equation (10) and the second equation in (5). Differentiating equation (11) gives

$$\dot{\boldsymbol{r}} = \sum_{n=0}^{\infty} \epsilon^n \frac{\partial \boldsymbol{r}^{(n)}}{\partial \boldsymbol{\phi}} [\boldsymbol{d} + \epsilon \boldsymbol{\Phi}(\boldsymbol{\phi}, \boldsymbol{r}(\boldsymbol{\phi}, \epsilon))]. \tag{12}$$

We want to write the right hand side as powers of ϵ.

In order to expand $\boldsymbol{\Phi}(\boldsymbol{\phi}, \boldsymbol{r}(\boldsymbol{\phi}, \epsilon))$ we need to use the multivariate form of the Taylor expansion

$$\boldsymbol{\Phi}(\boldsymbol{\phi}, \boldsymbol{r} + \boldsymbol{w}) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial^n \boldsymbol{\Phi}}{\partial \boldsymbol{r}^n}(\boldsymbol{\phi}, \boldsymbol{r}) \boldsymbol{w}^n, \tag{13}$$

where $\boldsymbol{r} = (r_1, r_2)^T$, $\boldsymbol{w} = (w_1, w_2)^T$ and

$$\frac{\partial^n \boldsymbol{\Phi}}{\partial \boldsymbol{r}^n}(\boldsymbol{\phi}, \boldsymbol{r}) \boldsymbol{w}^n = \left[\sum_{j_n=1}^{2} \cdots \sum_{j_1=1}^{2} \frac{\partial^n \Phi_i}{\partial r_{j_n} \cdots \partial r_{j_1}} (\boldsymbol{\phi}, \boldsymbol{r}) w_{j_1} \cdots w_{j_n} \right]_{i=1,2}. \tag{14}$$

Expanding $\boldsymbol{\Phi}(\boldsymbol{\phi}, \boldsymbol{r}(\boldsymbol{\phi}, \epsilon))$ using equation (13) and equation (14) gives

$$\begin{aligned} \boldsymbol{\Phi}\left[\boldsymbol{\phi}, \boldsymbol{r}^{(0)} + \sum_{s=1}^{\infty} \epsilon^s \boldsymbol{r}^{(s)}\right] &= \boldsymbol{\Phi}(\boldsymbol{\phi}, \boldsymbol{r}^{(0)}) + \sum_{n=1}^{\infty} \frac{1}{n!} \left[\sum_{j_n=1}^{2} \cdots \sum_{j_1=1}^{2} \frac{\partial^n \boldsymbol{\Phi}}{\partial r_{j_n} \cdots \partial r_{j_1}} \right. \\ & \left. \left[\sum_{s_1=1}^{\infty} \epsilon^{s_1} r_{j_1}^{(s_1)}\right] \cdots \left[\sum_{s_n=1}^{2} \epsilon^{s_n} r_{j_n}^{(s_n)}\right] \right]. \end{aligned} \tag{15}$$

It is not hard to show by an inductive argument that

$$\left[\sum_{s_1=1}^{\infty} \epsilon^{s_1} r_{j_1}^{(s_1)}\right] \cdots \left[\sum_{s_n=1}^{\infty} \epsilon^{s_n} r_{j_n}^{(s_n)}\right] = \sum_{s=n}^{\infty} \epsilon^s \left[\sum_{s_1+\cdots+s_n=s} r_{j_1}^{(s_1)} \cdots r_{j_n}^{(s_n)} \right], \tag{16}$$

where $j_1, \cdots, j_n = 1, 2$. Then equation (15) can be rewritten as

$$\begin{aligned} &\boldsymbol{\Phi}\left[\boldsymbol{\phi}, \boldsymbol{r}^{(0)} + \sum_{s=1}^{\infty} \epsilon^s \boldsymbol{r}^{(s)}\right] \\ &= \boldsymbol{\Phi}(\boldsymbol{\phi}, \boldsymbol{r}^{(0)}) + \sum_{n=1}^{\infty} \sum_{s=n}^{\infty} \epsilon^s \left[\sum_{s_1+\cdots+s_n=s} \frac{1}{n!} \frac{\partial^n \boldsymbol{\Phi}}{\partial \boldsymbol{r}^n} (\boldsymbol{\phi}, \boldsymbol{r}^{(0)}) \boldsymbol{r}^{(s_1)} \cdots \boldsymbol{r}^{(s_n)} \right]. \end{aligned} \tag{17}$$

To further expand equation (17) we make use of the following identity

$$\sum_{n=p}^{\infty} \sum_{s=n}^{\infty} \epsilon^s \boldsymbol{A}(n, s) = \sum_{n=p}^{\infty} \epsilon^n \left[\sum_{s=p}^{n} \boldsymbol{A}(s, n) \right], \tag{18}$$

where p is a nonnegative integer. Then equation (17) becomes

$$\begin{aligned} &\boldsymbol{\Phi}\left[\boldsymbol{\phi}, \boldsymbol{r}^{(0)} + \sum_{s=1}^{\infty} \epsilon^s \boldsymbol{r}^{(s)}\right] \\ &= \boldsymbol{\Phi}(\boldsymbol{\phi}, \boldsymbol{r}^{(0)}) + \sum_{n=1}^{\infty} \epsilon^n \sum_{k=1}^{n} \sum_{n_1+\cdots+n_k=n} \frac{1}{k!} \frac{\partial^k \boldsymbol{\Phi}}{\partial \boldsymbol{r}^k} (\boldsymbol{\phi}, \boldsymbol{r}^{(0)}) \boldsymbol{r}^{(n_1)} \cdots \boldsymbol{r}^{(n_k)}. \end{aligned} \tag{19}$$

In the next reduction the following identities are used:

$$\begin{aligned} \sum_{i=0}^{\infty} \sum_{n=1}^{\infty} \epsilon^{i+n+1} \boldsymbol{A}(i, n) &= \sum_{i=2}^{\infty} \epsilon^i \left[\sum_{n=0}^{i-2} \boldsymbol{A}(n, i-n-1) \right] \\ \sum_{i=p}^{\infty} \epsilon^{i+1} \boldsymbol{A}(i) &= \sum_{i=p+1}^{\infty} \epsilon^i \boldsymbol{A}(i-1). \end{aligned} \tag{20}$$

Inserting equation (19) into equation (12) and using (20)

$$
\begin{aligned}
\dot{\boldsymbol{r}} &= \sum_{i=0}^{\infty} \epsilon^i \frac{\partial \boldsymbol{r}^{(i)}}{\partial \boldsymbol{\phi}} \boldsymbol{d} + \sum_{i=0}^{\infty} \epsilon^{i+1} \frac{\partial \boldsymbol{r}^{(i)}}{\partial \boldsymbol{\phi}} \boldsymbol{\Phi}(\boldsymbol{\phi}, \boldsymbol{r}^{(0)}) \\
&\quad + \sum_{i=0}^{\infty} \sum_{n=1}^{\infty} \epsilon^{i+n+1} \frac{\partial \boldsymbol{r}^{(i)}}{\partial \boldsymbol{\phi}} \sum_{k=1}^{n} \sum_{n_1+\cdots+n_k=n} \frac{1}{k!} \frac{\partial^k \boldsymbol{\Phi}}{\partial \boldsymbol{r}^k} (\boldsymbol{\phi}, \boldsymbol{r}^{(0)}) \boldsymbol{r}^{(n_1)} \cdots \boldsymbol{r}^{(n_k)} \\
&= \frac{\partial \boldsymbol{r}^{(0)}}{\partial \boldsymbol{\phi}} \boldsymbol{d} + \epsilon \left[\frac{\partial \boldsymbol{r}^{(1)}}{\partial \boldsymbol{\phi}} \boldsymbol{d} + \frac{\partial \boldsymbol{r}^{(0)}}{\partial \boldsymbol{\phi}} \boldsymbol{\Phi}(\boldsymbol{\phi}, \boldsymbol{r}^{(0)}) \right] \\
&\quad + \sum_{i=2}^{\infty} \epsilon^i \left[\frac{\partial \boldsymbol{r}^{(i)}}{\partial \boldsymbol{\phi}} \boldsymbol{d} + \frac{\partial \boldsymbol{r}^{(i-1)}}{\partial \boldsymbol{\phi}} \boldsymbol{\Phi}(\phi, \boldsymbol{r}^{(0)}) \right. \\
&\quad \left. + \sum_{n=0}^{i-2} \frac{\partial \boldsymbol{r}^{(n)}}{\partial \boldsymbol{\phi}} \sum_{k=1}^{i-n-1} \sum_{n_1+\cdots+n_k=i-n-1} \frac{1}{k!} \frac{\partial^k \boldsymbol{\Phi}}{\partial \boldsymbol{r}^k} (\boldsymbol{\phi}, \boldsymbol{r}^{(0)}) \boldsymbol{r}^{(n_1)} \cdots \boldsymbol{r}^{(n_k)} \right].
\end{aligned}
\tag{21}
$$

Now we expand the right hand side of the second equation in (5) and get, by similar manipulations,

$$
\epsilon \boldsymbol{R}(\boldsymbol{\phi}, \boldsymbol{r}) = \epsilon \boldsymbol{R}(\boldsymbol{\phi}, \boldsymbol{r}^{(0)}) + \sum_{i=2}^{\infty} \epsilon^i \sum_{k=1}^{i-1} \sum_{n_1+\cdots+n_k=i-1} \frac{1}{k!} \frac{\partial^k \boldsymbol{R}}{\partial \boldsymbol{r}^k} \boldsymbol{r}^{(n_1)} \cdots \boldsymbol{r}^{(n_k)}. \tag{22}
$$

Equations (21) and (22) and rearranging gives the following sequence of partial differential equations

$$
\begin{aligned}
\frac{\partial \boldsymbol{r}^{(0)}}{\partial \boldsymbol{\phi}} \boldsymbol{d} &= \boldsymbol{0} \\
\frac{\partial \boldsymbol{r}^{(1)}}{\partial \boldsymbol{\phi}} \boldsymbol{d} &= \boldsymbol{R}(\boldsymbol{\phi}, \boldsymbol{r}^{(0)}) - \frac{\partial \boldsymbol{r}^{(0)}}{\partial \boldsymbol{\phi}} \boldsymbol{\Phi}(\boldsymbol{\phi}, \boldsymbol{r}^{(0)}) \\
\frac{\partial \boldsymbol{r}^{(i)}}{\partial \boldsymbol{\phi}} \boldsymbol{d} &= \sum_{k=1}^{i-1} \sum_{n_1+\cdots+n_k=i-1} \frac{1}{k!} \frac{\partial^k \boldsymbol{R}}{\partial \boldsymbol{r}^k} (\boldsymbol{\phi}, \boldsymbol{r}^{(0)}) \boldsymbol{r}^{(n_1)} \cdots \boldsymbol{r}^{(n_k)} \\
&\quad - \sum_{n=0}^{i-2} \frac{\partial \boldsymbol{r}^{(n)}}{\partial \boldsymbol{\phi}} \sum_{k=1}^{i-n-1} \sum_{n_1+\cdots+n_k=i-n-1} \frac{1}{k!} \frac{\partial^k \boldsymbol{\Phi}}{\partial \boldsymbol{r}^k} (\boldsymbol{\phi}, \boldsymbol{r}^{(0)}) \boldsymbol{r}^{(n_1)} \cdots \boldsymbol{r}^{(n_k)}
\end{aligned}
\tag{23}
$$

for $i \geq 2$.

To solve this system we will use a result from Diliberto *et al.* [10].

THEOREM 1. *A necessary and sufficient condition that there exist a doubly periodic solution of*

$$
\frac{\partial r}{\partial \phi_1} + \frac{\partial r}{\partial \phi_2} = G(\phi_1, \phi_2),
$$

with vector period $(2\pi/\mu_1, 2\pi/\mu_2)$ *in* (ϕ_1, ϕ_2) *is that*

$$
\int_0^{2\pi/\mu_1} \int_0^{2\pi/\mu_2} G(\phi_1, \phi_2)\, \mathrm{d}\phi_1\, \mathrm{d}\phi_2 = 0
$$

where $G(\phi_1, \phi_2)$ *has vector period* $(2\pi/\mu_1, 2\pi/\mu_2)$.

The first equation in (23) implies that $\boldsymbol{r}^{(0)}$ is a constant and that $\partial \boldsymbol{r}^{(0)}/\partial \boldsymbol{\phi} = \mathbf{0}$. The second equation then becomes

$$\frac{\partial \boldsymbol{r}^{(1)}}{\partial \boldsymbol{\phi}} \boldsymbol{d} = \boldsymbol{R}(\boldsymbol{\phi}, \boldsymbol{r}^{(0)}). \tag{24}$$

Theorem 1 implies that $\boldsymbol{r}^{(0)}$ must satisfy

$$\boldsymbol{R}_0(\boldsymbol{r}^{(0)}) = \frac{\mu_1\mu_2}{4\pi^2} \int_0^{2\pi/\mu_1} \int_0^{2\pi/\mu_2} \boldsymbol{R}(\boldsymbol{\phi}, \boldsymbol{r}^{(0)}) \, \mathrm{d}\phi_1 \, \mathrm{d}\phi_2 = 0, \tag{25}$$

which is the first of Hale's [1] conditions that a periodic surface exist. Once $\boldsymbol{r}^{(0)}$ is fixed then $\boldsymbol{r}^{(1)}$ is computed from equation (24) with integration constants. To fix $\boldsymbol{r}^{(1)}$, integration constants must be determined by setting the average of the right hand side of

$$\frac{\partial \boldsymbol{r}^{(2)}}{\partial \boldsymbol{\phi}} \boldsymbol{d} = \frac{\partial \boldsymbol{R}}{\partial \boldsymbol{r}} \boldsymbol{r}^{(1)} - \frac{\partial \boldsymbol{r}^{(1)}}{\partial \boldsymbol{\phi}} \boldsymbol{\Phi}(\boldsymbol{\phi}, \boldsymbol{r}^{(0)}) + \frac{\partial \boldsymbol{r}^{(0)}}{\partial \boldsymbol{\phi}} \frac{\partial \boldsymbol{\Phi}}{\partial \boldsymbol{r}} \boldsymbol{r}^{(1)} \tag{26}$$

to zero and solving for the integration constants. In particular we solve for integration constants such that

$$\int_0^{2\pi/\mu_1} \int_0^{2\pi/\mu_2} \left[\frac{\partial \boldsymbol{R}}{\partial \boldsymbol{r}} (\boldsymbol{\phi}, \boldsymbol{r}^{(0)}) \boldsymbol{r}^{(1)}(\boldsymbol{\phi}) - \frac{\partial \boldsymbol{r}^{(1)}}{\partial \boldsymbol{\phi}} \boldsymbol{\Phi}(\boldsymbol{\phi}, \boldsymbol{r}^{(0)}) \right] \mathrm{d}\phi_1 \, \mathrm{d}\phi_2 = \mathbf{0}. \tag{27}$$

Since the integration constants are the leading terms of $\boldsymbol{r}^{(1)}(\boldsymbol{\phi})$ and do not appear in $\partial \boldsymbol{r}^{(1)}/\partial \boldsymbol{\phi}$, their evaluation depends only on the invertibility of the matrix

$$\int_0^{2\pi/\mu_1} \int_0^{2\pi/\mu_2} \frac{\partial \boldsymbol{R}}{\partial \boldsymbol{r}} (\boldsymbol{\phi}, \boldsymbol{r}^{(0)}) \, \mathrm{d}\phi_1 \, \mathrm{d}\phi_2. \tag{28}$$

Assuming the matrix defined by (28) is invertible, this process continues. In general to solve for $\boldsymbol{r}^{(i)}$ we solve the general equation at the i-th stage for a particular solution, introduce an integration constant and fix it by setting the average of the right hand side of the $(i+1)$-th general equation to 0. From the structure of equation (23) we can state

THEOREM 2. *If* $\boldsymbol{R}_0(\boldsymbol{r}^{(0)}) = \mathbf{0}$ *and* (28) *is nonsingular then* (23) *can be solved sequentially for* $\boldsymbol{r}^{(0}$, $\boldsymbol{r}^{(1)}, \cdots$, *each of vector period* $(2\pi/\mu_1, 2\pi/\mu_2)$.

In the next section we will apply this technique to equation (1). In order not to become overburdened with algebraic detail we will show the construction of terms up to order ϵ only.

3. Application

In order to apply the results of the previous section to equation (1) we will transform it by introducing $u_1 = z_1$, $u_2 = \dot{z}_1$, $w_1 = z_2$, $w_2 = \dot{z}_2$ and getting

$$\begin{aligned} \dot{u}_1 &= u_2 \\ \dot{u}_2 &= -\mu_1^2 u_1 + \epsilon(1 - u_1^2 - a w_1^2) u_2 \\ \dot{w}_1 &= w_2 \\ \dot{w}_2 &= -\mu_2^2 w_1 + \epsilon(1 - \alpha u_1^2 - w_1^2) w_2. \end{aligned} \tag{29}$$

Let $u_1 = r_1^{1/2}\sin\mu_1\phi_1$, $u_2 = \mu_1 r_1^{1/2}\cos\mu_1\phi_1$, $w_1 = r_2^{1/2}\sin\mu_2\phi_2$, $w_2 = \mu_2 r_2^{1/2}\cos\mu_2\phi_2$, where $r_1, r_2 \geq 0$. Then equation (29) can be rewritten in the form

$$\begin{aligned}\dot{\boldsymbol{\phi}} &= \boldsymbol{d} + \epsilon\boldsymbol{\Phi}(\boldsymbol{\phi}, \boldsymbol{r}),\\ \dot{\boldsymbol{r}} &= \epsilon\boldsymbol{R}(\boldsymbol{\phi}, \boldsymbol{r}),\end{aligned} \tag{30}$$

where $\boldsymbol{d} = (1,1)^T$, $\boldsymbol{r} = (r_1, r_2)^T$, $\boldsymbol{\phi} = (\phi_1, \phi_2)^T$ and

$$\begin{aligned}\Phi_1(\boldsymbol{\phi}, \boldsymbol{r}) &= -\left[\frac{1}{\mu_1}\right](\sin\mu_1\phi_1\cos\mu_1\phi_1 - r_1\sin^3\mu_1\phi_1\cos\mu_1\phi_1\\ &\quad - ar_2\sin\mu_1\phi_1\cos\mu_1\phi_1\sin^2\mu_2\phi_2),\\ \Phi_2(\boldsymbol{\phi}, \boldsymbol{r}) &= -\left[\frac{1}{\mu_1}\right](\sin\mu_2\phi_2\cos\mu_2\phi_2 - \alpha r_1\sin^2\mu_1\phi_1\sin\mu_2\phi_2\cos\mu_2\phi_2\\ &\quad - r_2\sin^3\mu_2\phi_2\cos\mu_2\phi_2),\\ R_1(\boldsymbol{\phi}, \boldsymbol{r}) &= 2r_1(\cos^2\mu_1\phi_1 - r_1\sin^2\mu_1\phi_1\cos^2\mu_1\phi_1 - ar_2\cos^2\mu_1\phi_1\sin^2\mu_2\phi_2),\\ R_2(\boldsymbol{\phi}, \boldsymbol{r}) &= 2r_2(\cos^2\mu_2\phi_2 - \alpha r_1\sin^2\mu_1\phi_1\cos^2\mu_2\phi_2 - r_2\sin^2\mu_2\phi_2\cos^2\mu_2\phi_2).\end{aligned} \tag{31}$$

The components of the average in equation (7) for equation (31) are

$$\begin{aligned}R_{01}(\boldsymbol{r}) &= r_1\left(1 - \frac{r_1}{4} - a\frac{r_2}{2}\right)\\ R_{02}(\boldsymbol{r}) &= r_2\left(1 - \alpha\frac{r_1}{2} - \frac{r_2}{4}\right).\end{aligned} \tag{32}$$

If we solve for $\boldsymbol{r}$ such that $R_{01}(\boldsymbol{r}) = R_{02}(\boldsymbol{r}) = 0$ we get four cases:

1. $r_1 = r_2 = 0$
2. $r_1 = 0, \quad r_2 = 4$
3. $r_1 = 4, \quad r_2 = 0$
4. $r_1 = \dfrac{4-8a}{1-4a\alpha}, \quad r_2 = \dfrac{4-8\alpha}{1-4a\alpha}.$

The first case represents the trivial origin solution. The next two represent separate limit cycles. Case 4 is the more interesting case of the periodic surface and it is with this case that we will be concerned. We must first assume that $1 - 4a\alpha$ is nonzero. For numerical reasons we will consider values of a and α that lead to asymptotically stable periodic surfaces. By applying the Routh–Hurwitz criteria, Cronin [17], to $\partial\boldsymbol{R}_0/\partial\boldsymbol{r}$ it can be shown that the points in case 4 are asymptotically stable provided

$$\frac{1-a-\alpha}{1-4a\alpha} > 0, \qquad \frac{(2a-1)(2\alpha-1)}{1-4a\alpha} > 0. \tag{33}$$

Since we must also have $r_1 > 0$, $r_2 > 0$ this implies that $0 < a < 1/2$, $0 < \alpha < 1/2$ is the appropriate region of interest.

We can now proceed to the expansion by applying the iteration in equations (23). We assume that the expansion is given in the form equation (11). From the discussion after equation (26) we know that

$$\boldsymbol{r}^{(0)} = \left[\frac{4-8\alpha}{1-4a\alpha}, \frac{4-8\alpha}{1-4a\alpha}\right]^T. \tag{34}$$

For the sake of simplicity we will let $r_1^{(0)}$, $r_2^{(0)}$ be the components of $\boldsymbol{r}^{(0)}$. Then equation (26) becomes

$$\begin{aligned}\frac{\partial r_1^{(1)}}{\partial\phi_1}+\frac{\partial r_1^{(1)}}{\partial\phi_2} &= 2(r_1^{(0)})\cos^2\mu_1\phi_1-2(r_1^{(0)})^2\sin^2\mu_1\phi_1\cos^2\mu_1\phi_1\\ &\quad -2a(r_1^{(0)})(r_2^{(0)})\cos^2\mu_1\phi_1\sin^2\mu_2\phi_2,\\ \frac{\partial r_2^{(1)}}{\partial\phi_1}+\frac{\partial r_2^{(1)}}{\partial\phi_2} &= 2(r_2^{(0)})cos^2\mu_2\phi_2-2\alpha(r_1^{(0)})(r_2^{(0)})\sin^2\mu_1\phi_1\cos^2\mu_2\phi_2\\ &\quad -2(r_2^{(0)})^2\sin^2\mu_2\phi_2\cos^2\mu_2\phi_2.\end{aligned}\tag{35}$$

By insertion of an assumed trigonometric form it is straightforward to show that $r_1^{(1)}$, $r_2^{(1)}$ are given by

$$\begin{aligned}r_1^{(1)}=A &+ \left[\frac{1}{2\mu_1}\right]\left[r_1^{(0)}-\frac{ar_1^{(0)}r_2^{(0)}}{2}\right]\sin 2\mu_1\phi_1\\ &+\left[\frac{1}{4\mu_1}\right]\left[\frac{(r_1^{(0)})^2}{4}\right]\sin 4\mu_1\phi_1\\ &+\left[\frac{1}{2\mu_2}\right]\left[\frac{ar_1^{(0)}r_2^{(0)}}{2}\right]\sin 2\mu_2\phi_2\\ &+\left[\frac{1}{2\mu_1+2\mu_2}\right]\left[\frac{ar_1^{(0)}r_2^{(0)}}{4}\right]\sin(2\mu_1\phi_1+2\mu_2\phi_2)\\ &+\left[\frac{1}{2\mu_1-2\mu_2}\right]\left[\frac{ar_1^{(0)}r_2^{(0)}}{4}\right]\sin(2\mu_1\phi_1-2\mu_2\phi_2),\end{aligned}\tag{36}$$

$$\begin{aligned}r_2^{(1)}=B &+ \left[\frac{1}{2\mu_1}\right]\left[\frac{\alpha r_1^{(0)}r_2(0)}{2}\right]\sin 2\mu_1\phi_1\\ &+\left[\frac{1}{2\mu_2}\right]\left[r_2^{(0)}-\frac{\alpha r_1^{(0)}r_2^{(0)}}{2}\right]\sin 2\mu_2\phi_2\\ &+\left[\frac{1}{4\mu_2}\right]\left[\frac{(r_2^{(0)})^2}{4}\right]\sin 4\mu_2\phi_2\\ &+\left[\frac{1}{2\mu_1+2\mu_2}\right]\left[\frac{\alpha r_1^{(0)}r_2^{(0)}}{4}\right]\sin(2\mu_1\phi_1+2\mu_2\phi_2)\\ &+\left[\frac{1}{2\mu_1-2\mu_2}\right]\left[\frac{\alpha r_1^{(0)}r_2^{(0)}}{4}\right]\sin(2\mu_1\phi_1-2\mu_2\phi_2).\end{aligned}$$

In order to determine the integration constants A and B we compute the right hand side of

$$\frac{\partial\boldsymbol{r}^{(2)}}{\partial\boldsymbol{\phi}}\boldsymbol{d}=\frac{\partial\boldsymbol{R}}{\partial\boldsymbol{r}}\boldsymbol{r}^{(1)}-\frac{\partial\boldsymbol{r}^{(1)}}{\partial\boldsymbol{\phi}}\boldsymbol{\Phi}(\boldsymbol{\phi},\boldsymbol{r}^{(0)}),\tag{37}$$

which when written in component form is

$$\begin{aligned}
\frac{\partial r_1^{(2)}}{\partial \phi_1} + \frac{\partial r_1^{(2)}}{\partial \phi_2} &= \frac{\partial R_1}{\partial r_1}(\boldsymbol{\phi}, \boldsymbol{r}^{(0)}) r_1^{(1)} + \frac{\partial R_1}{\partial r_2}(\boldsymbol{\phi}, \boldsymbol{r}^{(0)}) r_2^{(1)} \\
&\quad - \frac{\partial r_1^{(1)}}{\partial \phi_1}\Phi_1(\boldsymbol{\phi}, \boldsymbol{r}^{(0)}) - \frac{\partial r_1^{(0)}}{\partial \phi_2}\Phi_2(\boldsymbol{\phi}, \boldsymbol{r}^{(0)}), \\
\frac{\partial r_2^{(2)}}{\partial \phi_1} + \frac{\partial r_2^{(2)}}{\partial \phi_2} &= \frac{\partial R_2}{\partial r_1}(\boldsymbol{\phi}, \boldsymbol{r}^{(0)}) r_1^{(1)} + \frac{\partial R_2}{\partial r_2}(\boldsymbol{\phi}, \boldsymbol{r}^{(0)}) r_2^{(1)} \\
&\quad - \frac{\partial r_2^{(1)}}{\partial \phi_1}\Phi_1(\boldsymbol{\phi}, \boldsymbol{r}^{(0)}) - \frac{\partial r_2^{(1)}}{\partial \phi_2}\Phi(\boldsymbol{\phi}, \boldsymbol{r}^{(0)}).
\end{aligned} \tag{38}$$

Since each of the right hand sides contains a significant number of terms we only give the averaged equations for the right hand sides of equation (38).

$$\begin{aligned}
&\left[1 - \frac{r_1^{(0)}}{2} - \frac{a r_2^{(0)}}{2}\right] A - \left[\frac{a r_1^{(0)}}{2}\right] B = 0, \\
&\left[-\frac{\alpha r_2^{(0)}}{2}\right] A + \left[1 - \frac{\alpha r_1^{(0)}}{2} - \frac{r_2^{(0)}}{2}\right] B = 0.
\end{aligned} \tag{39}$$

By the choice of $r_1^{(0)}$ and $r_2^{(0)}$ we know that

$$\begin{aligned}
&1 - \frac{r_1^{(0)}}{4} - \frac{a r_2^{(0)}}{2} = 0, \\
&1 - \frac{\alpha r_1^{(0)}}{2} - \frac{r_2^{(0)}}{4} = 0.
\end{aligned} \tag{40}$$

Using equations (40), equations (39) reduce to

$$\begin{aligned}
&\left[-\frac{r_1^{(0)}}{4}\right] A + \left[-\frac{a r_1^{(0)}}{2}\right] B = 0, \\
&\left[-\frac{\alpha r_2^{(0)}}{2}\right] A + \left[-\frac{r_2^{(0)}}{4}\right] B = 0.
\end{aligned} \tag{41}$$

The determinant of this system is

$$\left[\frac{1}{16}\right] (r_1^{(0)} r_2^{(0)})(1 - 4a\alpha). \tag{42}$$

But by choice $r_1^{(0)}$, $r_2^{(0)}$ and $(1 - 4a\alpha)$ are all nonzero. This implies from equations (41) that $A = B = 0$. From equations (36) we have $r_1^{(1)}$ and $r_2^{(1)}$. Finally then to first order in ϵ we can write

$$\boldsymbol{r}(\boldsymbol{\phi}, \epsilon) = \boldsymbol{r}^{(0)} + \epsilon \boldsymbol{r}^{(1)}, \tag{43}$$

where $\boldsymbol{r}^{(0)}$ is given by equations (34) and $\boldsymbol{r}^{(1)} = (r_1^{(1)}, r_2^{(1)})^T$ where $r_1^{(1)}$ and $r_2^{(1)}$ are given by equations (36) with $A = B = 0$. The procedure can be continued in order to generate higher order terms by using a symbolic manipulation program.

4. Numerical Results

Given equation (43) we can investigate the behavior of the solutions on the torus and compare them with the numerical integration of equations (30) near the torus. All of the following integrations were performed on a personal computer with math coprocessor support. Double precision arithmetic was used in the integration routine, which was based on Gear's method [18]. To be specific we will consider the case with

$$\begin{aligned} \epsilon &= 0.05, \\ a &= 0.20, \\ \alpha &= 0.40, \\ \mu_1 &= 1.0, \\ \mu_2 &= 1.414. \end{aligned} \tag{44}$$

From equations (34) we have

$$\begin{aligned} r_{01} &= 3.53, \\ r_{02} &= 1.18. \end{aligned} \tag{45}$$

For the sake of notation let

$$\hat{\boldsymbol{\Phi}}(\boldsymbol{\phi}, \epsilon) = \boldsymbol{\Phi}(\boldsymbol{\phi}, \boldsymbol{r}(\boldsymbol{\phi}, \epsilon)), \tag{46}$$

where $\boldsymbol{r}(\boldsymbol{\phi}, \epsilon)$ comes from equation (43) and Φ is defined by the first two equations of (31). The flow on the torus is generated by integrating the phase equation

$$\dot{\phi} = \boldsymbol{d} + \epsilon\hat{\boldsymbol{\Phi}}(\boldsymbol{\phi}, \epsilon) \tag{47}$$

and substituting the results into equation (43).

Figures 1 (a) and 1 (b) show two pictures of the torus. These pictures are drawn to show that integrating the full system of equations given by equations (30) and integrating equation (47) on the torus lead to graphically indistinguishable tori. In Figure 1 (a) the full system (30), (31) was integrated with an initial condition on the appropriate torus. In Figure 1 (b) system (47) was integrated and the radial values were computed from equation (43). The same initial conditions were used for all of the figures. In particular the integration conditions were $\phi_1 = 0.0$, $\phi_2 = 0.0$, $r_1 = 3.53$, $r_2 = 1.18$, $\delta t = 0.2$, # steps = 1500.

Since the quality of the approximation cannot be determined graphically a more detailed error study must be performed. Figure 2 shows what the phases and amplitudes look like after integrating on the torus. Figures 2(a) and 2(b) show the phases from the integration of equation (47) on the torus with Figures 2(c) and 2(d) generated from equation (43). The phases are linear as would be suggested from equation (47). The amplitudes for $\boldsymbol{r}$ oscillate about $r_1 = 3.53$ and $r_2 = 1.18$. The amplitudes from integrating equations (30) and (31) are not shown since they are nearly equivalent to Figures 2 (a) through 2 (d) and would add no new information. One advantage of using a representation of the torus in all of the computations lay in the amount of computing time saved. Integrating equations (47) and (43) on the torus took one fifth the time of integrating the equations (30) and (31). This was experienced with all of the numerical calculations.

In order to study the periodicities superimposed on the linearities shown in Figure 2 a least squares program was used to subtract the trend lines. Figure 3 shows the periodicities superimposed

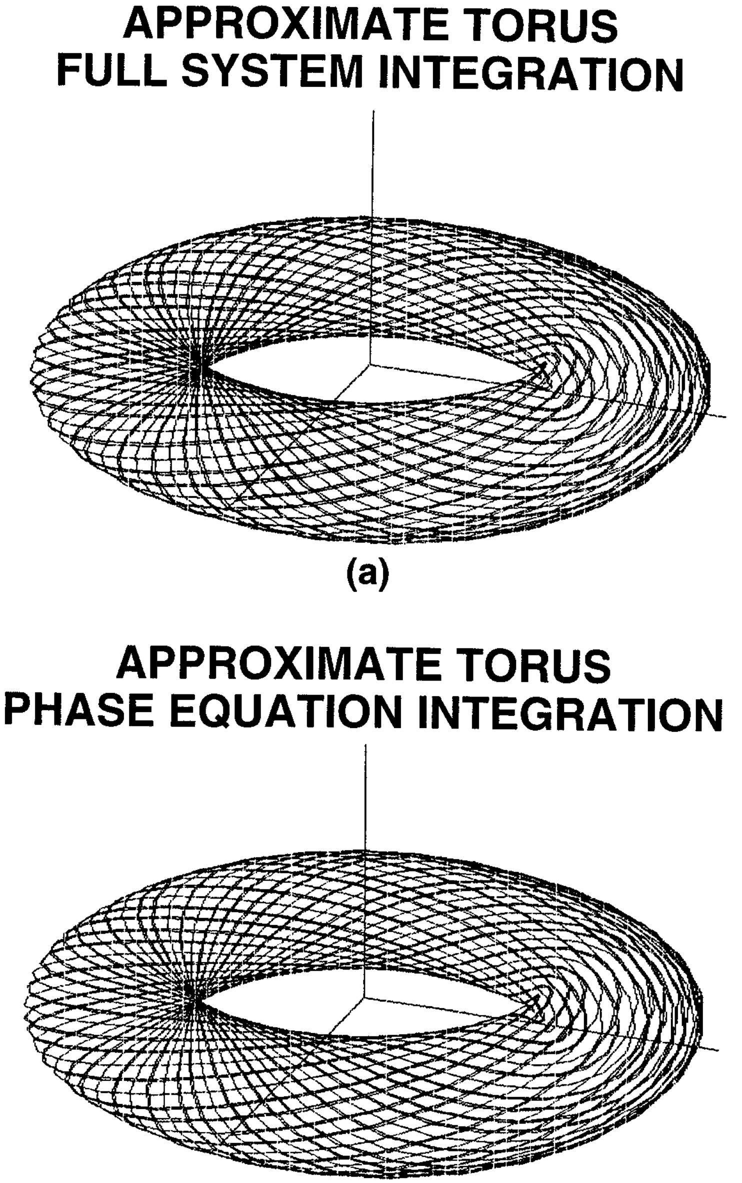

Fig. 1. Trajectories on the torus: (a) full system, (b) phase equations only. Initial conditions: $\phi_1 = 0$, $\phi_2 = 0$, $r_1 = 3.53$, $r_2 = 1.18$, integration step size = 0.2, number of steps = 1500.

upon the trends shown in Figure 2. Again the periodicities superimposed on the trends from integrating equation (3) are not shown since they are graphically similar to Figure 3.

Figure 4, however, shows some of the details of the differences of integrating the full system (30) near the torus and integrating equation (47) on the torus. These figures show the absolute error differences between system (30) and equations (47), (43). Table I shows the maximum errors given in Figure 4. This table shows that the errors for 1500 steps are an order of magnitude better than

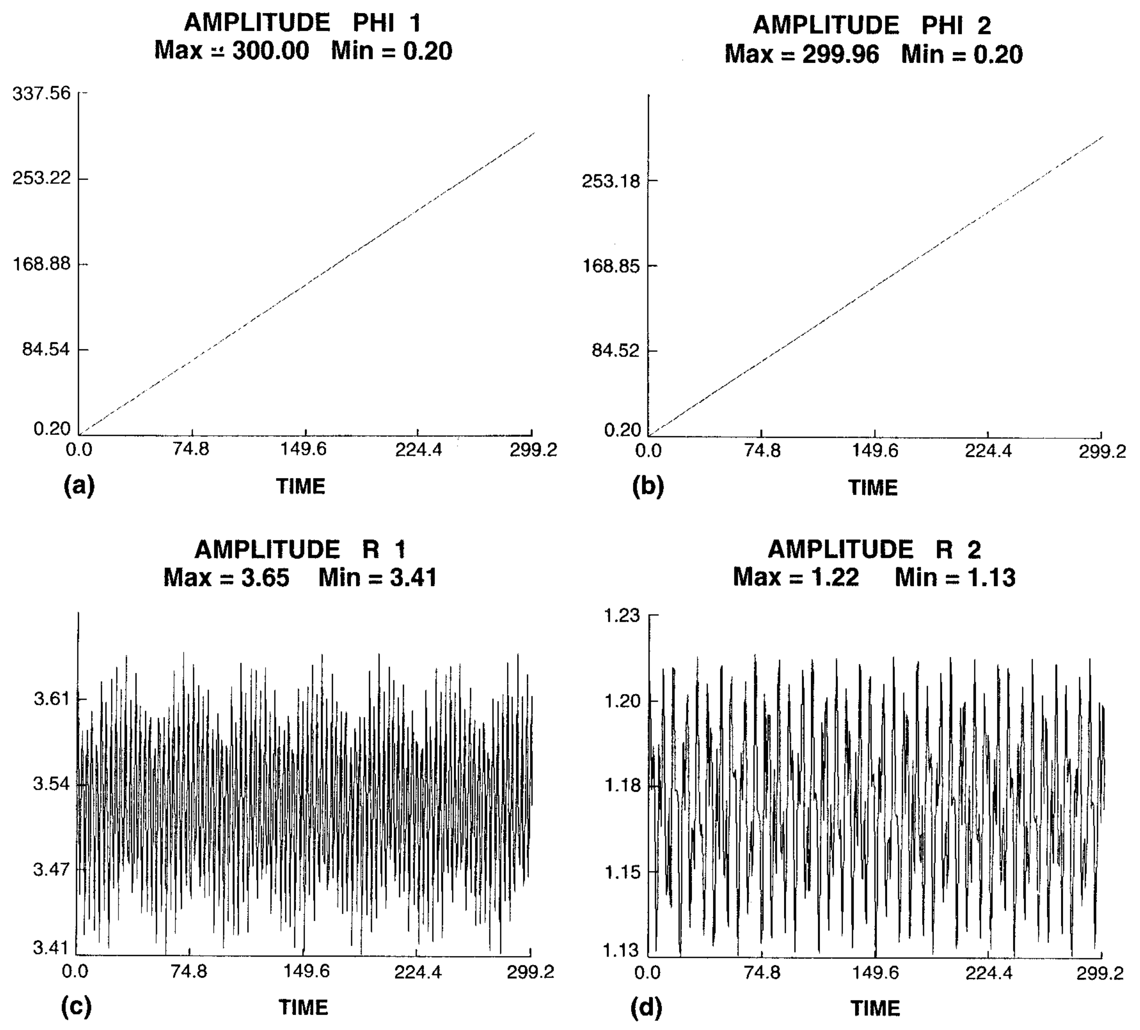

Fig. 2. Integrating phase equations on the torus: (a) and (b) show the phase amplitudes, (c) and (d) show the radial amplitudes. The initial conditions are the same as Figure 1.

the order of the perturbation parameter, $\epsilon = 0.05$.

An extended error comparison between the phase and radial equations in system (30) and those of the approximation given by equations (47) and (43) are given in Table II. Both systems were integrated for 10,000 steps and the maximum absolute and relative errors up to the step number shown are given in the appropriate columns. For example under the ϕ_1 columns the maximum absolute error and relative error between the ϕ_1 phase equation in system (30) and that in equation (47) are shown. At 1000 steps of $\delta t = 0.2$ the maximum absolute error of all steps up to 1000 is 2.0E–3 and the maximum relative error is 1.0E–5. Note that from step 5000 to 10,000 the ϕ_1 maximum errors do not change. The absolute error can be interpreted as a measure of the number of digits to the right of the decimal that compare favorably and the relative error can be interpreted as a measure of the overall number of digits that compare favorably beginning with the leading digit. For example at step 9000 $\phi_1 = 1799.878$ from system (30) and $\phi_1 = 1799.909$ from equation (47).

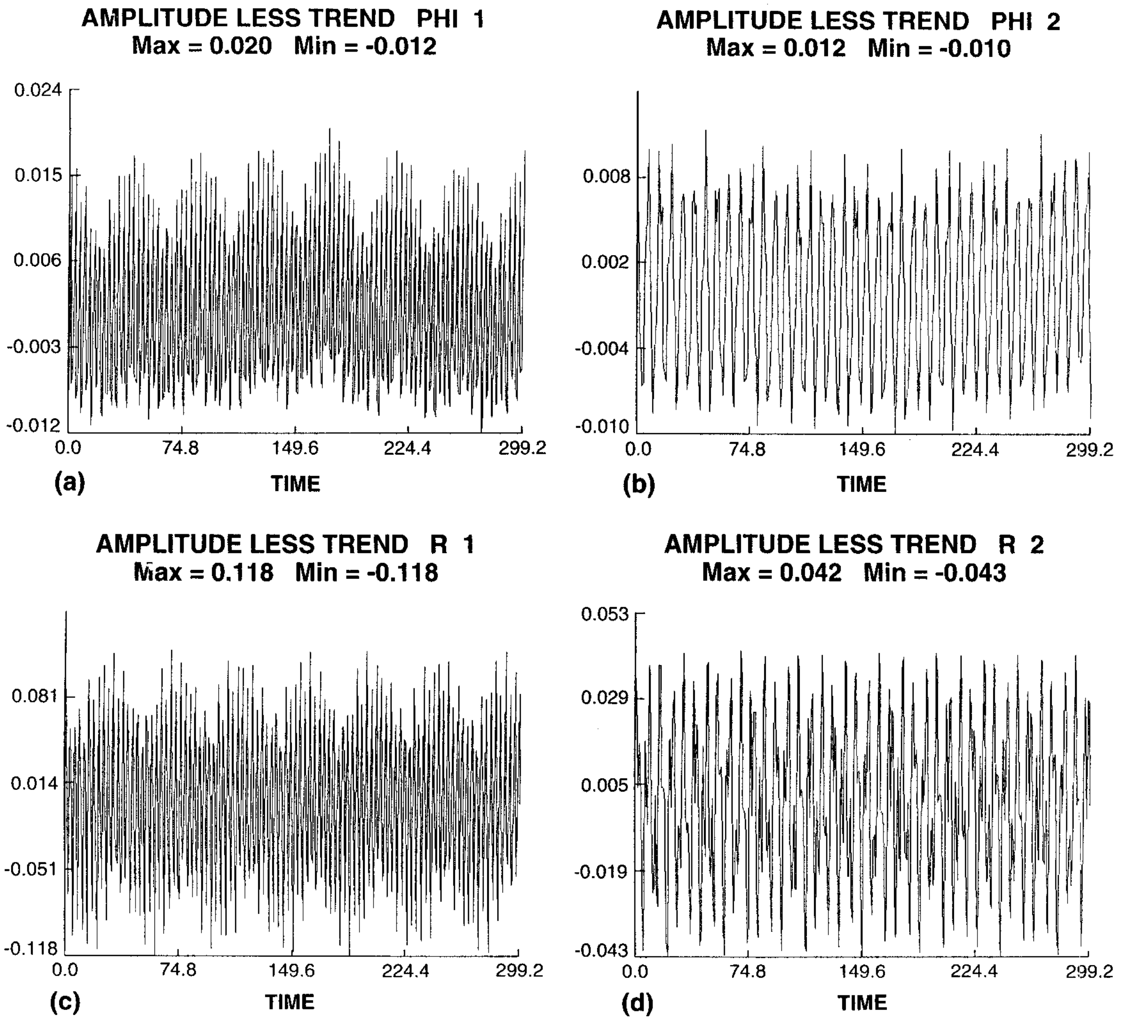

Fig. 3. These figures show in greater detail the periodicities superimposed on the linear trends of the amplitudes shown in Figure 2.

TABLE I

Maximum errors between integration of the full van der Pol system and the phase equations on the torus for 1500 steps

Equation	Max. Abs. Error
ϕ_1	0.0026
ϕ_2	0.0055
r_1	0.0042
r_2	0.0016

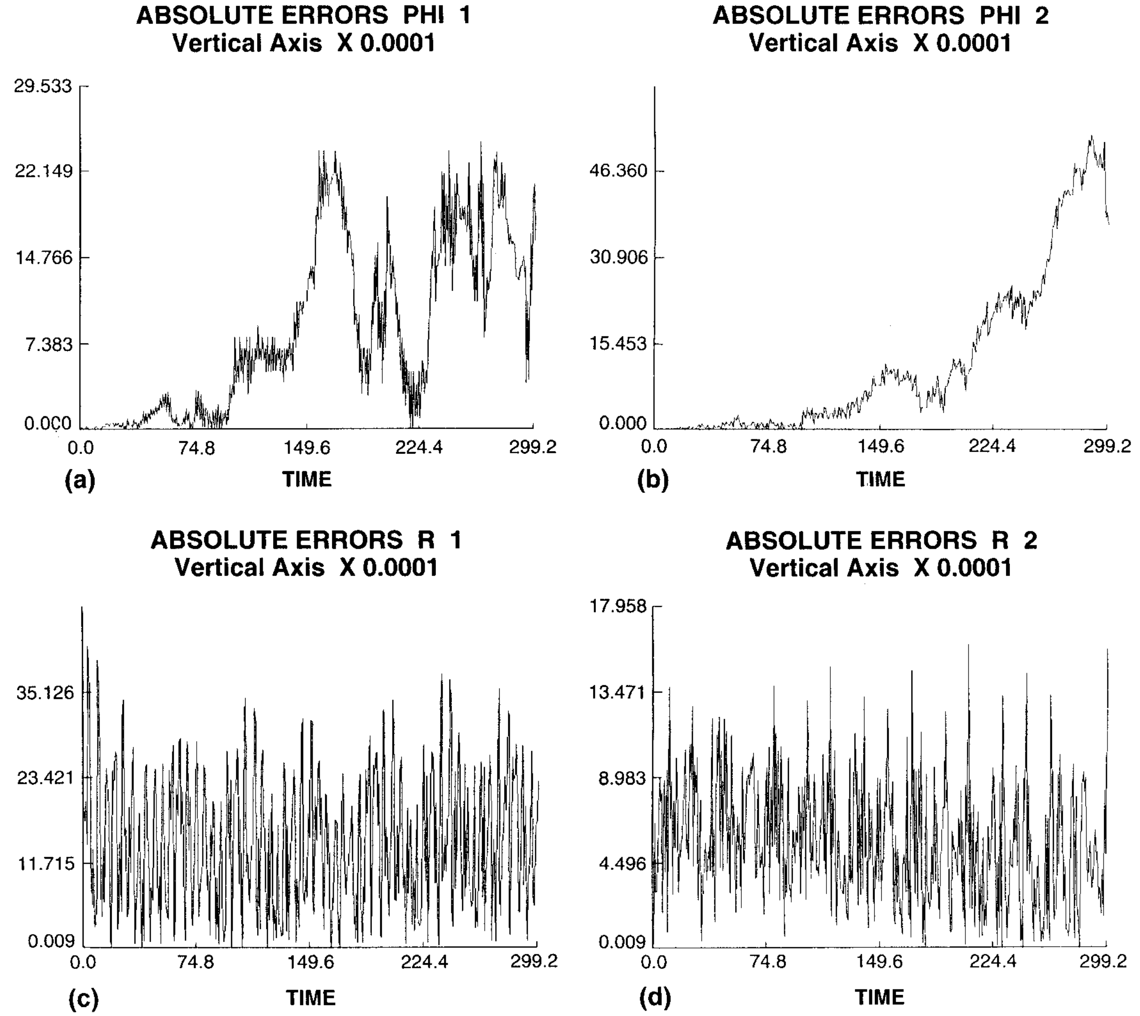

Fig. 4. Absolute errors between integrating the full system and the phase equations on the torus. (a) and (b) show the phase differences where (c) and (d) show radial differences. Initial conditions as in Figure 1.

The absolute error here is 3.1E–2 which is less than the maximum given in Table II as 6.7E–2. This maximum relative error says that a little more than one digit after the decimal compares exactly. The maximum relative error of 7.2E–5 at step 9000 of ϕ_1 says that somewhat fewer than 5 digits compare exactly. This is reflected in the example above. The same analysis applies to the other columns. In order to interpret r_1 and r_2 note that in magnitude they have one digit to the left of the decimal as is clear from equations (43) and (45). Although Table II shows that both types of errors grow as the number of steps increase it also shows that the absolute errors are all approximately of the order $\epsilon = 0.05$ throughout the 10,000 steps.

Another measure of the quality of the approximate formula for the torus is whether it reproduces the periodicities experienced by integrating the full system (30) near the torus. The comparison is shown graphically in Figures 5 and 6. In Figure 5 the log scaled amplitudes of the periodograms of each of the angular and radial components for system (3) with the trends removed is shown. In

TABLE II

Long term accrued errors between the van der Pol system and the phase equations on the torus for 10,000 steps

	ϕ_1		ϕ_2	
Steps	Abs. Err.	Rel. Err.	Abs. Err.	Rel. Err.
1000	2.0E–3	1.0E–5	5.0E–4	2.7E–6
2000	1.0E–2	3.6E–5	2.4E–3	7.0E–6
3000	1.3E–2	3.6E–5	2.4E–3	7.0E–6
4000	5.0E–2	6.3E–5	6.1E–3	8.5E–6
5000	6.7E–2	7.2E–5	7.3E–3	8.5E–6
6000	6.7E–2	7.2E–5	1.1E–2	9.5E–6
7000	6.7E–2	7.2E–5	2.1E–2	1.5E–5
8000	6.7E–2	7.2E–5	2.3E–2	1.5E–5
9000	6.7E–2	7.2E–5	4.2E–2	2.3E–5
10000	6.7E–2	7.2E–5	5.4E–2	2.7E–5
	r_1		r_2	
Steps	Abs. Err.	Rel. Err.	Abs. Err.	Rel. Err.
1000	2.9E–3	8.1E–4	1.4E–3	1.2E–3
2000	4.4E–3	1.2E–3	1.7E–3	1.4E–3
3000	4.4E–3	1.2E–3	1.8E–3	1.5E–3
4000	8.6E–3	2.4E–3	2.1E–3	1.8E–3
5000	1.2E–2	3.3E–3	2.2E–3	2.0E–3
6000	1.3E–2	3.8E–3	2.4E–3	2.3E–3
7000	1.3E–2	3.8E–3	2.7E–3	2.4E–3
8000	1.3E–2	3.8E–3	2.8E–3	3.4E–3
9000	1.3E–2	3.8E–3	4.0E–3	4.4E–3
10000	1.3E–2	3.8E–3	5.3E–3	4.4E–3

Figure 6 the comparable graphs are shown for the approximation given by equations (47) and (43). It is clear from these figures that the dominant peaks are reproduced by the approximation formula. How close they reproduce the peaks is shown in Table III. This table is organized into three sets of columns under each of the angular and radial components. The columns under the full system title are taken from Figure 5. Those under the approximate columns are taken from Figure 6. In the first column of each set the frequency (in Hertz) of the peak is given. In the second column the letter ‘s’ means that the peak is strong, or has greater than one order of magnitude of amplitude. A ‘w’ means that the peak is weak or has less than an order of magnitude in amplitude. The third column in each set gives the log of the amplitude of the periodogram for the given frequency. The same columns are given under the approximate set. Finally the last column shows the best estimate of the theoretical radial frequency, ω, in radians/sec, associated with the peak frequency. Only those radial frequencies that are present in equations (36) are identified. Notice that they represent the strongest peaks. The weaker peaks are represented by higher harmonics that do not appear in equations (36). The radial frequencies are related to the frequencies in Hertz (f) by $\omega = 2\pi f$. A glance at Table III shows that the approximate formula for the torus from equations (47) and (43) not only reproduces the strong peaks but also reproduce the log amplitudes of the components. Differences occur in the weak peaks or those with log amplitudes near 0. Table III shows that the first order approximation

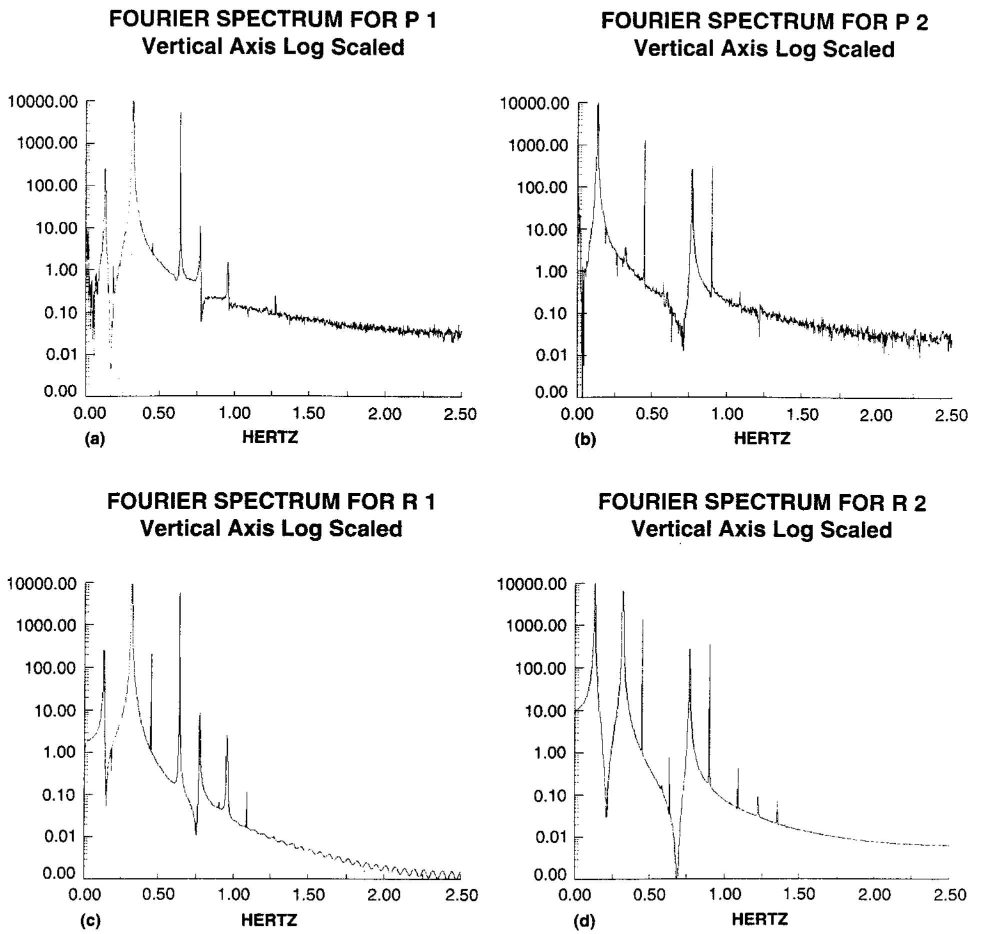

Fig. 5. Spectral plots of the amplitudes of the phase and radial equations for the full system with the linear trends removed. The initial conditions are as in Figure 1.

equations (36) capture the essential periodicities on the torus.

Another interesting question of the flow on the torus involves the rotation number, Coddington and Levinson [19]. In Figure 1 take the x-axis toward the viewer, the y-axis toward the right and the z-axis vertical. Then let parallels be the curves on the torus cut by planes parallel to the xy-plane. Let a meridian be the curve cut by the xz-plane and the torus. The rotation number can be thought of as the average rotation about the meridian for one trip around the parallel. To compute the rotation number begin by defining the quotient

$$\frac{\mathrm{d}\phi_2}{\mathrm{d}\phi_1} = \frac{1 + \epsilon\hat{\Phi}_2(\phi_1, \phi_2, \epsilon)}{1 + \epsilon\hat{\Phi}_1(\phi_1, \phi_2, \epsilon)}, \tag{48}$$

where $\hat{\Phi}_1$, $\hat{\Phi}_2$ are the components of equation (46). Note that $\hat{\Phi}_1$, $\hat{\Phi}_2$ are periodic in ϕ_1 with period $P_1 = 2\pi/\mu_1$ and in ϕ_2 with period $P_2 = 2\pi/\mu_2$. Let the solution of (48) be written as

$$\phi_2 = \phi_2(\phi_1, \eta), \tag{49}$$

TABLE III

Frequencies for the periodicities superimposed on the linear trends for each of the phase and radial components

ϕ_1

Full System			Approximate			Theoretical
Freq.(Hz)	Qual.	Log. Amp.	Freq.	Qual.	Log. Amp.	Rad./Sec.
0.1367	s	2.41	0.1333	s	2.43	$2\mu_2 - 2\mu_1$
0.3167	s	4.00	0.3167	s	4.00	$2\mu_1$
0.4500	w	0.64	–	–	–	$2\mu_2$
0.6400	s	3.76	0.6400	s	3.77	$4\mu_1$
0.7667	s	1.04	0.7700	s	1.08	$2\mu_2 + 2\mu_1$
0.9533	s	0.19	0.9567	w	0.19	
1.2733	w	-0.61	–	–	–	

ϕ_2

Full System			Approximate			Theoretical
Freq.(Hz)	Qual.	Log. Amp.	Freq.	Qual.	Log. Amp.	Rad./Sec.
0.1333	s	4.00	0.1333	s	4.00	$2\mu_2 - 2\mu_1$
0.4467	s	3.12	0.4500	s	3.12	$2\mu_2$
0.7667	s	2.44	0.7700	s	2.41	$2\mu_2 + 2\mu_1$
0.9000	s	2.50	0.9000	s	2.50	$4\mu_2$

r_1

Full System			Approximate			Theoretical
Freq.(Hz)	Qual.	Log. Amp.	Freq.	Qual.	Log. Amp.	Rad./Sec.
0.1300	s	2.41	0.1300	s	2.41	$2\mu_2 - 2\mu_1$
0.3167	s	4.00	0.3167	s	4.00	$2\mu_1$
0.6367	s	3.76	0.6367	s	3.76	$4\mu_1$
0.7700	s	0.94	0.7700	s	0.93	$2\mu_2 + 2\mu_1$
0.9533	s	0.40	0.9533	s	-0.42	
–	–	–	1.2733	s	-0.27	
1.0867	w	-0.93	–	–	–	

r_2

Full System			Approximate			Theoretical
Freq.(Hz)	Qual.	Log. Amp.	Freq.	Qual.	Log. Amp.	Rad./Sec.
0.1333	s	4.00	0.1333	s	4.00	$2\mu_2 - 2\mu_1$
0.3167	s	3.83	0.3167	s	3.83	$2\mu_1$
0.4500	s	3.14	0.4500	s	3.13	$2\mu_2$
0.6367	s	-0.11	0.6367	s	0.08	$4\mu_1$
0.7700	s	2.45	0.7700	s	2.45	$2\mu_2 + 2\mu_1$
0.9000	s	2.54	0.9000	s	2.53	$4\mu_2$
–	–	–	0.9533	w	-0.46	
–	–	–	1.0333	w	-0.68	
1.0867	w	-0.37	1.0867	w	-0.67	
1.2200	w	-1.04	–	–	–	
1.3500	w	-1.13	–	–	–	

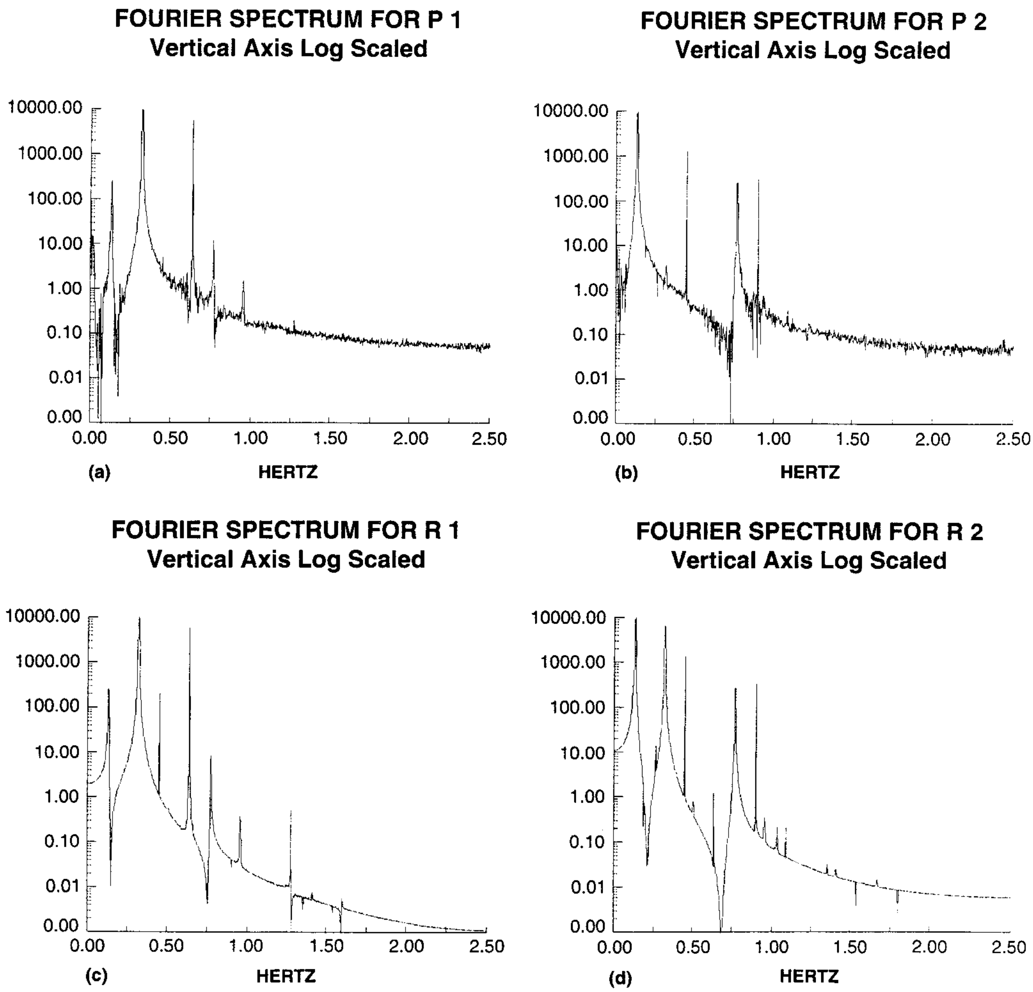

Fig. 6. Spectral plots of the amplitudes of the phase and radial equations for the approximate system with linear trends removed. The initial conditions are as in Figure 1.

where

$$\phi_2(0, \eta_0) = \eta_0, \tag{50}$$

and $0 \leq \eta_0 \leq P_2$. Define

$$\eta_n = \phi_2(nP_1, \eta_{n-1}) \tag{51}$$

and the rotation number as

$$\rho = lim_{n\to\infty} \frac{\eta_n}{nP_2}. \tag{52}$$

Using (50) one can estimate the rotation number. In particular if $\epsilon = 0$ in equation (48) then

$$\frac{\mathrm{d}\phi_2}{\mathrm{d}\phi_1} = 1. \tag{53}$$

Integrating over n periods P_1 gives

$$\eta_n = 2\pi n/\mu_1. \tag{54}$$

Then from equation (52) $\rho = \mu_2/\mu_1$. Using conditions (44) an integration of equation (48) for 100 traverses around the torus yields $\rho = 1.413941$ which approximates $\mu_2/\mu_1 = 1.414$ (see equation (44)). The difference between the computed result and 1.414 comes about because $\epsilon > 0$ in equation (48).

5. Conclusions

This paper has shown that there is a straightforward algorithm that unites a perturbation series with averaging to generate approximate tori for autonomous coupled nonlinear oscillators of the form of equation (3). The perturbation series requires the solution of a sequence of partial differential equations. These are shown to be easily solvable by trigonometric series of multiple angles. In the application of the result of two nonlinearly coupled van der Pol oscillators the computation showed that a significant time savings is possible for the study of flows on the torus by the availability of the approximate torus representation. The comparison of the integration of the full system of coupled oscillators in angular standard form and the approximate phase and radial equations on the torus showed that the accumulation of error remained less than the order of the approximation for long integration periods. In particular for an approximation of the first order in the perturbation parameter the errors in the comparisons of the phase and radial equations maintained themselves to the same order. Thus for a perturbation parameter $\epsilon = 0.05$ the errors in the phase and radial equations were all approximately 0.05 or less. The periodicities superimposed on the background trends are essentially maintained between the full and approximate systems. This is shown by a spectral comparison that shows that not only are the predominant peaks recovered by the approximation formula but also the strength of each peak is comparable. Finally the rotation of the flow on the torus compares favorably with that predicted.

6. Acknowledgements

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